ADVANCES IN MATHEMATICS 14, 335–345 (1974)

A Kallman–Rota Inequality for Nearly Euclidean Spaces*

JOHN A. R. HOLBROOK

Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, Canada

1. INTRODUCTION

R. R. Kallman and G.-C. Rota have proved (see [5]) that

$$\| Ax \|^2 \leqslant 4 \| x \| \| A^2 x \| \tag{1}$$

whenever A is the infinitesimal generator of a strongly continuous semigroup of contractions on a Banach space $(X, \| \cdot \|)$ and x, Ax are in the domain of A. An interesting example is obtained when the space is $(L^p[0, \infty), \| \cdot \|_p)$ and A is differentiation. In fact, an old result of G. H. Hardy, J. E. Littlewood, and G. Pólya (see [2, p. 187]) applies to the case $p = 2$, and asserts that

$$\| f' \|_2^2 \leqslant 2 \| f \|_2 \| f'' \|_2 \tag{2}$$

for any function f on $[0, \infty)$ such that $f, f', f'' \in L^2[0, \infty)$. T. Kato (see [6]) has "explained" the improvement in the constant by proving that the constant 4 in (1) may be replaced by 2 whenever X is a *Hilbert space.*

This suggests that the Kallman–Rota inequality may be improved for any space in which the geometry is sufficiently nearly Euclidean (in some appropriate sense). In what follows (see Theorem 9) we obtain some results of this nature. Our estimates may be applied, for example, to show that the best constant $K(p)$ in the inequality

$$\| f' \|_p^2 \leqslant K(p) \| f \|_p \| f'' \|_p \quad (f, f' f'' \in L^p[0, \infty)) \tag{3}$$

approaches 2 as $p \to 2$; for information of this kind, see (8) and the remarks following Theorem 9. Rather than assuming that A is an

* This work was supported in part by the National Research Council of Canada under Grant No. A8745.

infinitesimal generator as in the Kallman–Rota theorem, we shall work (except in Section 5) with the slightly more general class of *dissipative* operators. Unfortunately our methods do not appear to yield the Kallman–Rota constant of 4 when applied to a general normed space; they seem especially suited to spaces with a very nearly Euclidean geometry.

2. Some Geometric Notions

We shall always suppose that $(X, \|\cdot\|)$ is a real or complex normed linear space.

Definition 1. For any normed space X, let

$$a(X) = \sup\{(\| x + y \|^2 + \| x - y \|^2)/(2 \| x \|^2 + 2 \| y \|^2) : x, y \in X\}.$$

Definition 2. If $x, y \in X$, we say that *y is at an obtuse angle from x* provided $\operatorname{Re} \phi_x(y) \leqslant 0$ whenever ϕ_x is an element of the dual space X^* such that $\| \phi_x \| = 1$ and $\phi_x(x) = \| x \|$.

Note that a linear operator A with domain $D(A) \subset X$ (and range in X) is *dissipative* exactly when Ax is at an obtuse angle from x for all $x \in D(A)$. Note also that if X is an *inner product space*, y is at an obtuse angle from x if and only if $\operatorname{Re}(y, x) \leqslant 0$.

Definition 3. For any normed space X, let

$$b(X) = \sup\{\| x + y \|/\| x - y \| : x, y \in X, \text{ and } y \text{ is at an obtuse angle from } x\}.$$

We next collect some simple facts about the notions we have just introduced.

Proposition 4. *For any $x, y \in X$,*

$$\begin{aligned} a(X)^{-1}[2 \| x \|^2 + 2 \| y \|^2] &\leqslant \| x + y \|^2 + \| x - y \|^2 \\ &\leqslant a(X)[2 \| x \|^2 + 2 \| y \|^2]. \end{aligned} \tag{4}$$

For any X, $1 \leqslant a(X) \leqslant 2$, and the extreme case $a(X) = 1$ holds if and only if X is an inner product space.

Proof. The second inequality of (4) is immediate, and the first follows from it upon replacing x by $(x + y)/2$ and y by $(x - y)/2$.

For any, $x, y \in X$,

$$\| x + y \|^2 + \| x - y\|^2 \leqslant 2(\| x \| + \| y \|)^2 = 2 \| x \|^2 + 2 \| y \|^2 + 4 \| x \| \| y \|,$$

and since $2 \| x \| \| y \| \leqslant \| x \|^2 + \| y \|^2$, it is clear that $a(X) \leqslant 2$. Certainly, $a(X) \geqslant 1$, since we may have $x = y$.

Finally, it is a well-known result of P. Jordan and J. von Neumann (see [4]) that $(X, \| \cdot \|)$ is an inner product space if and only if the parallelogram law holds for $\| \cdot \|$. Q.E.D.

Proposition 4 suggests that $a(X)$ somehow measures the extent to which the normed space X has a Euclidean geometry. We shall therefore say that X is *C-Euclidean* whenever $a(X) \leqslant C$.

PROPOSITION 5. *If $X = L^p$, i.e., X is the L^p-space over some measure space, $1 < p < \infty$, then X is $2^{|1-2/p|}$-Euclidean.*

Proof. This will be a simple consequence of the Clarkson inequalities (see J. A. Clarkson [1, Theorem 2]). Note that in what follows, we use the Hölder inequality for the inner product in $\mathbb{R}^2$ several times.

Consider first the case $p \geqslant 2$. For $f, g \in L^p$, we have

$$\begin{aligned} &\| f + g \|_p^2 + \| f - g \|_p^2 \\ &\quad \leqslant (\| f + g \|_p^p + \| f - g \|_p^p)^{2/p} (1 + 1)^{1-2/p}; \end{aligned}$$

by the Clarkson inequality, this is dominated by

$$\begin{aligned} &(2[\| f \|_p^q + \| g \|_p^q]^{p-1})^{2/p}\, 2^{(1-2/p)} \\ &= 2[\| f \|_p^q + \| g \|_p^q]^{2/p} \\ &\leqslant 2[(\| f \|_p^2 + \| g \|_p^2)^{q/2} (1 + 1)^{1-q/2}]^{2/q} \\ &= 2^{(1-2/p)}[2 \| f \|_p^2 + 2 \| g \|_p^2]. \end{aligned}$$

Similarly, for $1 < p \leqslant 2$, we need only argue that

$$\begin{aligned} \| f + g \|_p^2 + \| f - g \|_p^2 &\leqslant (\| f + g \|_p^q + \| f - g \|_p^q)^{2/q} (1 + 1)^{1-2/q} \\ &\leqslant (2[\| f \|_p^p + \| g \|_p^p]^{q-1})^{2/q}\, 2^{(1-2/q)} \\ &= 2[\| f \|_p^p + \| g \|_p^p]^{2/p} \\ &\leqslant 2[(\| f \|_p^2 + \| g \|_p^2)^{p/2} (1 + 1)^{1-p/2}]^{2/p} \\ &= 2^{2/p-1}[2 \| f \|_p^2 + 2\| g \|_p^2]. \end{aligned}$$

Q.E.D.

PROPOSITION 6. *For any X, $1 \leqslant b(X) \leqslant 3$, and $b(X) = 1$ if X is an inner product space.*

Proof. If y is at an obtuse angle from x, then

$$\| x - y \| \geqslant \operatorname{Re} \phi_x(x - y) = \| x \| - \operatorname{Re} \phi_x(y) \geqslant \| x \|.$$

Hence, if $\| y \| \leqslant 2 \| x \|$, we have $\| x + y \| \leqslant 3 \| x \| \leqslant 3 \| x - y \|$. But if $\| y \| > 2 \| x \|$,

$$\| x + y \| \leqslant \| x \| + \| y \| \leqslant (3/2) \| y \|,$$

while

$$\| x - y \| \geqslant \| y \| - \| x \| \geqslant (1/2) \| y \|.$$

In any case, $\| x + y \| / \| x - y \| \leqslant 3$, so that $b(X) \leqslant 3$. Since we may have $y = 0$, $b(X) \geqslant 1$.

If y is at an obtuse angle from x in an inner product space, then $\operatorname{Re}(y, x) \leqslant 0$, so that $\| x + y \|^2 - \| x - y \|^2 = 4 \operatorname{Re}(y, x) \leqslant 0$. Q.E.D.

While we do not need such a result here, it seems a reasonable conjecture that the extreme case $b(X) = 1$ occurs *only* if X is an inner product space. It *is* important for our present purposes to know that $b(X)$ is close to 1 when X is nearly Euclidean, i.e., when $a(X)$ is close to 1. We shall obtain a result of this sort in the next section.

3. An Estimate for $b(X)$ when X is C-Euclidean

We need the following technical lemma.

LEMMA 7. *If $(X, \| \cdot \|)$ is C-Euclidean and $x, y \in X$ are linearly independent, there is an inner product $(\cdot, \cdot)$ on the 2-dimensional real subspace S spanned by x and y such that the corresponding norm $| \cdot |$ $(= (\cdot, \cdot)^{1/2})$ satisfies:*

$$| x | = \| x \|$$

and

$$M^{-1} | s | \leqslant \| s \| \leqslant M | s | \quad (\text{all } s \in S),$$

where $M = (1 + 19(C - 1))^{1/2}$.

Proof. Consider first the case where $M \geqslant \sqrt{2}$, i.e., $\rho = C - 1 \geqslant 1/19$. It is well-known (see, for example, the more general result in F. John [3]) that for any norm $\| \cdot \|$ on a real 2-dimensional space S, there is an inner product norm $\rho(\cdot)$ on S such that

$$\rho(s) \leqslant \| s \| \leqslant \sqrt{2}\, \rho(s) \quad (\text{all } s \in S).$$

If we define $| s | = (\| x \| / \rho(x))\, \rho(s)$, we clearly obtain an inner product norm $| \cdot |$ such that $| x | = \| x \|$ and $(1/\sqrt{2})\, | s | \leqslant \| s \| \leqslant \sqrt{2}\, | s |$ (all $s \in S$). Since we are assuming $M \geqslant \sqrt{2}$, this is all we need.

Turning to the case where $\rho = C - 1 < 1/19$, we set $u = x/\| x \|$, and let v be some vector in S such that $\| v \| = 1$ and $\| u + v \| = \| u - v \|$. We shall define $| s |$ for $s \in S$ by setting

$$| \alpha u + \beta v |^2 = \alpha^2 + \beta^2 \; (\alpha, \beta \in \mathbb{R}).$$

Clearly $| \cdot |$ is an inner product norm on S, and $| x | = \| x \|$. It remains to show that

$$M^{-1} | s | \leqslant \| s \| \leqslant M \, | s | \quad (\text{all } s \in S).$$

By homogeneity we may assume that s has one of the forms $\pm u \pm tv$, $\pm v \pm tu$, where $t \in [0, 1]$. Since the properties $\| u \| = \| v \| = 1$, $\| u + v \| = \| u - v \|$ are independent of the order and sign of u, v, we shall assume $s = u + tv$, $t \in [0, 1]$.

Consider the function $f(t) = \| u + tv \|^2 - \| u - tv \|^2$; certainly $f\colon [0, 1] \to \mathbb{R}$ is continuous and $f(0) = f(1) = 0$. Since $(X, \| \cdot \|)$ is C-Euclidean, we may observe that for t, $r \in [0, 1]$,

$$\begin{aligned}
f(t) + f(r) &= (\| u + tv \|^2 + \| u + rv \|^2) - (\| u - tv \|^2 + \| u - rv \|^2) \\
&\leqslant (C/2)(\| 2u + tv + rv \|^2 + \| tv - rv \|^2) \\
&\quad - (1/C2)(\| 2u - tv - rv \|^2 + \| - tv + rv \|^2) \\
&= 2C\left(\left\| u + \left(\frac{t + r}{2}\right) v \right\|^2 + \left(\frac{t - r}{2}\right)^2\right) \\
&\quad - (2/C)\left[\left\| u - \left(\frac{t + r}{2}\right) v \right\|^2 + \left(\frac{t - r}{2}\right)^2\right] \\
&= 2f\left(\frac{t + r}{2}\right) + 2(C - 1)\left[\left\| u + \left(\frac{t + r}{2}\right) v \right\|^2 + \left(\frac{t - r}{2}\right)^2\right] \\
&\quad + 2(1 - C^{-1})\left[\left\| u - \left(\frac{t + r}{2}\right) v \right\|^2 + \left(\frac{t - r}{2}\right)^2\right].
\end{aligned}$$

Hence $f(t) + f(r) - 2f((t + r)/2) \leqslant \epsilon$, where

$$\begin{aligned}\epsilon &= 2(C - 1)[2^2 + 1/4] + 2(C - 1)[2^2 + 1/4]\\ &= 17(C - 1).\end{aligned}$$

If we replace v by $-v$ above, f changes sign, so that we must, in fact, have

$$\left|f(t) + f(r) - 2f\left(\frac{t + r}{2}\right)\right| \leqslant \epsilon. \tag{5}$$

Let f attain its maximum value at $t_0 \in [0, 1]$. If $t_0 \in [0, 1/2]$, let $t = 2t_0$ and $r = 0$ in (5) to obtain $|f(t) - 2f(t_0)| \leqslant \epsilon$. It follows that $2f(t_0) \leqslant f(t) + \epsilon \leqslant f(t_0) + \epsilon$, so that $f(t_0) \leqslant \epsilon$. If $t_0 \in [1/2, 1]$, let $t = 1 - 2(1 - t_0)$ and $r = 1$ in (5) to obtain, again, $|f(t) - 2f(t_0)| \leqslant \epsilon$, and to conclude that $f(t_0) \leqslant \epsilon$ in this case also. Since the same sort of argument can be carried out for the minimum value of f, we see that $|f(t)| \leqslant \epsilon$ for all $t \in [0, 1]$.

Now calling again on the C-Euclidean property, we see that

$$\begin{aligned}\| u + tv \|^2 &= 1/2(f(t) + (\| u + tv \|^2 + \| u - tv \|^2))\\ &\leqslant 1/2(\epsilon + C[2 \| u \|^2 + 2 \| tv \|^2]) = \epsilon/2 + C(1 + t^2)\\ &\leqslant (C + \epsilon/2)(1 + t^2) = (C + \epsilon/2)| u + tv |^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\| u + tv \|^2 &\geqslant 1/2(- \epsilon + C^{-1}[2 \| u \|^2 + 2 \| tv \|^2])\\ &= -\epsilon/2 + C^{-1}(1 + t^2) \geqslant (C^{-1} - \epsilon/2)| u + tv |^2.\end{aligned}$$

It remains to show that $M^2 \geqslant (C + \epsilon/2)$ and $M^{-2} \leqslant (C^{-1} - \epsilon/2)$. Certainly,

$$C + \epsilon/2 = C + (17/2)(C - 1) = 1 + (19/2)(C - 1) \leqslant 1 + 19(C - 1) = M^2.$$

On the other hand,

$$\begin{aligned}C^{-1} - \epsilon/2 &= (1 + \rho)^{-1} - (17/2)\rho \geqslant 1 - \rho - (17/2)\rho\\ &= 1 - (19/2)\rho \geqslant (1 + 19\rho)^{-1}(= M^{-2}),\end{aligned}$$

since $\rho \leqslant 1/19$. Q.E.D.

THEOREM 8. *If $(X, \|\cdot\|)$ is C-Euclidean, then*

$$b(X) \leqslant \mu(\mu^{1/2} + (\mu - 1)^{1/2}),$$

where $\mu = 1 + 19(C - 1)$.

Proof. Given $x, y \in X$ with y at an obtuse angle from x, we must establish the appropriate upper bound for $\| x + y \|/\| x - y \|$. We may assume that $\| x \| = 1$. We may also suppose that x and y are linearly independent, since if $y = zx$ for some scalar z, we have $0 \geqslant \operatorname{Re} \phi_x(y) = \operatorname{Re}(z \| x \|) = \operatorname{Re} z$ (here we choose some functional ϕ_x as in Definition 2) and hence

$$\| x + y \|/\| x - y \| = | 1 + z |/| 1 - z | \leqslant 1.$$

By the lemma above, we may introduce an inner product $(\cdot, \cdot)$ with corresponding norm $|\cdot|$ on the real subspace S spanned by x and y in such a way that

$$| x | = \| x \| \ (= 1),$$

and

$$M^{-1} | s | \leqslant \| s \| \leqslant M | s |, \quad (\text{all } s \in S),$$

where $M = (1 + 19(C - 1))^{1/2}\ (= \mu^{1/2})$.

Let $w = y/| y |$, and note that for any $t \geqslant 0$,

$$\begin{aligned} | x - tw | &\geqslant M^{-1} \| x - tw \| \geqslant M^{-1} \operatorname{Re} \phi_x(x - tw) \\ &= M^{-1}(1 - (t/| y |) \operatorname{Re} \phi_x(y)) \geqslant M^{-1}. \end{aligned}$$

Now, for each $t > 0$,

$$\begin{aligned} M^{-2} \leqslant | x - tw |^2 &= | x |^2 - 2t(x, w) + t^2 | w |^2 \\ &= 1 - 2t(x, w) + t^2, \end{aligned}$$

so that

$$(x, w) \leqslant 1/2[t^{-1}(1 - M^{-2}) + t],$$

and, minimizing with respect to t, we obtain

$$(x, w) \leqslant (1 - M^{-2})^{1/2}.$$

If $r = (x, w)$, we have for any $t \geqslant 0$,

$$| x + tw |^2/| x - tw |^2 = (1 + 2tr + t^2)/(1 - 2tr + t^2),$$

so that if $r < 0$, $| x + tw | \leqslant | x - tw |$. On the other hand, it is easily verified that if $0 \leqslant r < 1$,

$$\max_{t \geqslant 0} [(1 + 2tr + t^2)/(1 - 2tr + t^2)] = (1 + r)/(1 - r),$$

so that, in any case,

$$\frac{| x + tw |}{| x - tw |} \leqslant \left[\frac{1 + (1 - M^{-2})^{1/2}}{1 - (1 - M^{-2})^{1/2}}\right]^{1/2} = M + (M^2 - 1)^{1/2}.$$

Hence

$$\frac{\| x + y \|}{\| x - y \|} \leqslant \frac{M \, | x + y |}{M^{-1} \, | x - y |} \leqslant M^2(M + (M^2 - 1)^{1/2}). \qquad \text{Q.E.D.}$$

4. Dissipative Operators on a C-Euclidean Space

The following theorem reveals an explicit connection between the constant in the inequalities of Kallman–Rota–Kato type and the geometry of the underlying normed space.

THEOREM 9. *If A is a dissipative linear operator with domain $D(A)$ in a normed space X, then*

$$\| Ax \|^2 \leqslant a(X)(1 + b(X)^2) \, \| x \| \, \| A^2 x \|, \tag{6}$$

whenever x, $Ax \in D(A)$. If X is C-Euclidean,

$$\| Ax \|^2 \leqslant C(1 + \mu^2(2\mu - 1 + 2(\mu^2 - \mu)^{1/2})) \| x \| \, \| A^2 x \|, \tag{7}$$

where $\mu = 1 + 19(C - 1)$.

Remarks. If X is an inner product space, then, as we have noted in Section 2, it is immediate that $a(X) = b(X) = 1$, and we obtain from (6) the Kato inequality:

$$\| Ax \|^2 \leqslant 2 \, \| x \| \, \| A^2 x \|.$$

The inequality (7) follows from (6) via Theorem 8. The constant in (7) certainly approaches 2 as C tends to 1, but its exact form is probably of little significance.

Since (Proposition 5) L^p-spaces are $2^{|1-2/p|}$-Euclidean, (7) provides an estimate which, while decidedly awkward, does show that the constants $K(p)$ of inequality (3) approach the Hardy–Littlewood–Polya value 2 as $p \to 2$.

Proof of (6). Since A is dissipative, Av is at an obtuse angle from v for any $v \in D(A)$, so that $\|(1 + A)v\| \leqslant b(X)\|(1 - A)v\|$. Setting $v = (1 + A)x$, we obtain $\|(1 + A^2)x\| \leqslant b(X)\| x - A^2x \|$. Since $2Ax = (1 + A)^2x - (x + A^2x)$. we may argue that

$$\begin{aligned} 2\| Ax \| &\leqslant \|(1 + A)^2x\| + \| x + A^2x \| \\ &\leqslant b(X)\| x - A^2x \| + \| x + A^2x \| \\ &\leqslant (b(X)^2 + 1)^{1/2}(\| x - A^2x \|^2 + \| x + A^2x \|^2)^{1/2} \\ &\leqslant (1 + b(X)^2)^{1/2}\, (a(X)[2 \| x \|^2 + 2 \| A^2x \|^2])^{1/2}. \end{aligned}$$

Now, for any $t \geqslant 0$, tA is certainly dissipative along with A; replacing A by tA above, we easily obtain

$$\| Ax \|^2 \leqslant \tfrac{1}{2}\, a(X)(1 + b(X)^2)(t^{-2} \| x \|^2 + t^2 \| A^2x \|^2).$$

Minimizing with respect to t results in (6). Q.E.D.

5. Contractive Semigroups on L^p-Spaces

Under some additional hypotheses, the techniques of Theorem 9 may be combined with standard results on interpolation between L^p-spaces to obtain more precise inequalities. The semigroup of translations $([T(t)f](x) = f(x + t))$ is strongly continuous on each $L^p[0, \infty)$, $1 \leqslant p < \infty$, and the infinitesimal generator represents differentiation in this case. Hence, one consequence of the following theorem is that the constants $K(p)$ occurring in (3) satisfy the inequality

$$K(p) \leqslant 2^{|1-2/p|}(1 + 9^{|1-2/p|}). \tag{8}$$

Theorem 10. *Suppose that $\{T(t): t \geqslant 0\}$ is a semigroup of operators on the (equivalence classes of) measurable (real or complex) functions over a fixed measure space M. Suppose further that, for each p such that $1 \leqslant$*

$p < \infty$, the semigroup is contractive and strongly continuous when restricted to $L^p(M)$. Then, if A_p is the infinitesimal generator in $L^p(M)$,

$$\| A_p f \|_p^2 \leqslant 2^{|1-2/p|}(1 + 9^{|1-2/p|})\| f \|_p \| A_p^2 f \|_p , \tag{9}$$

whenever f and $A_p f$ are in the domain of A_p .

Proof. For each function (equivalence class) f in the domain D_p of A_p , $A_p f$ is the limit, in the $L^p(M)$-sense, of the sequence $n(T(1/n)f - f)$. This makes it clear that $A_p f = A_{p'} f$ whenever $1 \leqslant p, p' < \infty$ and $f \in D_p \cap D_{p'}$. From the general theory of semigroups we know that $(1 - A_p)$ has a (contractive) inverse defined everywhere in $L^p(M)$, and that

$$(1 - A_p)^{-1} f = \int_0^\infty e^{-t} T(t) f \, dt,$$

for any $f \in L^p(M)$. This formula makes it easy to see that $(1 - A_p)^{-1}$ is independent of p in the sense that $(1 - A_p)^{-1} f = (1 - A_{p'})^{-1} f$ whenever $f \in L^p(M) \cap L^{p'}(M)$. The same is therefore true of the operators V_p defined by

$$V_p f = (1 + A_p)(1 - A_p)^{-1} f \quad (f \in L^p(M)).$$

Note that if $g = (1 - A_p)^{-1} f$,

$$\| V_p f \|_p / \| f \|_p = \|(1 + A_p) g \|_p / \|(1 - A_p) g \|_p ,$$

so that $\| V_p \| \leqslant b(L^p(M))$, since A_p is dissipative. Thus, each V_p is a bounded operator on $L^p(M)$, and, by Proposition 6, $\| V_p \| \leqslant 3$ and $\| V_2 \| = 1$.

Consider a value of p in the range $2 \leqslant p < \infty$, and let $p < p' < \infty$. By the Riesz convexity theorem we have

$$\begin{aligned} \log \| V_p \| &\leqslant \frac{1/2 - 1/p}{1/2 - 1/p'} \log \| V_{p'} \| + \frac{1/p - 1/p'}{1/2 - 1/p'} \log \| V_2 \| \\ &\leqslant \frac{1/2 - 1/p}{1/2 - 1/p'} \log 3, \end{aligned}$$

and, letting $p' \to \infty$, we obtain

$$\log \| V_p \| \leqslant (1 - 2/p) \log 3.$$

Similarly, interpolation between $L^1(M)$ and $L^2(M)$ yields

$$\log \| V_p \| \leqslant (2/p - 1) \log 3$$

when $1 \leqslant p \leqslant 2$, so that

$$\| V_p \| \leqslant 3^{|1-2/p|} \; (1 \leqslant p < \infty).$$

Now, for any $g \in D_p$,

$$\begin{aligned} \|(1 + A_p) g \|_p &= \| V_p(1 - A_p) g \|_p \\ &\leqslant \| V_p \| \cdot \|(1 - A_p) g \|_p \leqslant 3^{|1-2/p|} \|(1 - A_p) g \|_p , \end{aligned}$$

and, by using this estimate in place of

$$\|(1 + A)v \| \leqslant b(X) \| (1 - A)v \|$$

in the proof of (6), we obtain

$$\| A_p f \|_p^2 \leqslant a(L^p(M))(1 + (3^{|1-2/p|})^2) \| f \|_p \| A_p^2 f \|_p .$$

The inequality (9) then follows by Proposition 5. Q.E.D.

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